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On Solutions of Certain Self-Adjoint Differential Equations of Fourth Order

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In this paper the author considers the differential equation $[r(x)y'']'' - p(x)y = 0$, where $r(x)$ and $p(x)$ are continuous, $r(x) > 0$, and $p(x) \neq 0$ on an interval $[\epsilon, \infty)$, $\epsilon > 0$. The cases for which $p(x)$ is positive and $p(x)$ is negative on $[\epsilon, \infty)$ are treated separately.

In the first part of the paper it is assumed that $p(x)$ is positive on $[\epsilon, \infty)$. A study is made of the general properties of solutions of the equation, particularly, oscillatory solutions. Special emphasis is given the case when all oscillatory solutions are bounded. A rather simple representation for all bounded oscillatory solutions is given.

The function $p(x)$ is assumed to be negative in the second part of the paper. Some of Marko Svec's results on the behavior of oscillatory solutions are extended to this more general equation. Also two theorems analogus to the Bôcher-Osgood theorem for second-order equations are given. Finally, the author gives some necessary conditions that certain types of nonoscillatory solutions exist.

INTRODUCTION

Consider the self-adjoint linear differential equation

$$[r(x)\,y'']'' + p(x)\,y = 0, \tag{1.1}$$

where $r(x)$ and $p(x)$ are continuous and $r(x) > 0$ on the interval $[\epsilon, \infty)$ for some $\epsilon > 0$. As pointed out by Leighton and Nehari [7], the behavior of the solutions of (1.1) is radically different according as $p(x)$ is positive or negative on $[\epsilon, \infty)$. This behavior is well known in the analogous second-order equation

$$(r(x)\,y')' + p(x)\,y = 0. \tag{1.2}$$

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We shall then consider separately the cases $p(x) > 0$ and $p(x) < 0$ on $[\epsilon, \infty)$. This distinction will be denoted by writing equation (1.1) in the forms

$$[r(x)\, y''(x)]'' - p(x)\, y = 0; \qquad r(x) > 0, \quad p(x) > 0, \tag{1.3}$$

and

$$[r(x)\, y''(x)]'' + p(x)\, y = 0; \qquad r(x) > 0, \quad p(x) > 0. \tag{1.4}$$

The case when $p(x)$ may vanish on $[\epsilon, \infty)$ will not be discussed.

In this paper we shall study the behavior of oscillatory solutions of Eq. (1.3) and (1.4). An *oscillatory solution* of either Eq. (1.3) or (1.4) is a nontrivial solution which has unbounded zeros. A nontrivial solution which is not oscillatory is said to be *nonoscillatory*. If the equation has an oscillatory solution then the equation is said to be *oscillatory*. Otherwise, the equation is said to be *nonoscillatory*.

We now state two definitions which are frequently used in this paper. Two solutions are said to be *essentially different* provided they are not constant multiples of each other. A solution $y(x)$ that satisfies a property P is *essentially unique* provided any solution satisfying property P is a constant multiple of $y(x)$.

The first part of this paper is concerned with solutions of (1.3). This part is divided into three sections. The first section is a statement of some basic results already known. In the second section we give some new general properties of solutions of (1.3). We shall be primarily interested in those properties concerning oscillatory solutions. In the third section we restrict our attention to the case when the oscillatory solutions are bounded. Some of the properties discussed in the second section are refined. We shall also give a representation for every bounded oscillatory solution of (1.3).

The second part of this paper is devoted to the study of solutions of (1.4). In the first section of this part of the paper we extend to (1.4) many of Svec's results for the special case $r(x) \equiv 1$. We also give sufficient conditions on the coefficients concerning the boundedness of certain types of oscillatory solutions. The second section is a discussion of the case when (1.4) is known to be nonoscillatory. We give some properties of nonoscillatory solutions and some necessary conditions that certain types of nonoscillatory solutions exist.

The following result is fundamental. Its proof is given in [7].

LEMMA 1.1. *Let $u(x)$ and $v(x)$ be functions of class C' in (a, b), and let $v(x)$ be of constant sign in this interval. If $x = \alpha$ and $x = \beta$ $(a < \alpha < \beta < b)$ are consecutive zeros of $u(x)$, there then exists a constant k such that the function $u(x) - kv(x)$ has a double zero in (α, β).*

PART I

2. *Fundamental Notions*

The first part of this paper is devoted to the study of the equation

$$[r(x) y'']'' - p(x) y = 0, \tag{2.1}$$

where $r(x)$ and $p(x)$ are positive and continuous on $[\epsilon, \infty)$, $\epsilon > 0$. As a basis for our study of Eq. (2.1), we state the following four lemmas. These lemmas play an extremely important role in our studies. Their proofs may be found in the paper of Leighton and Nehari [7].

LEMMA 2.1. *If $y(x)$ is a solution of* (2.1) *and the values of y, y', y'', and $(ry'')'$ are nonnegative, but not all zero, for $x = a \geqslant \epsilon$, then the functions $y(x)$, $y'(x)$, $y''(x)$, and $[r(x) y''(x)]'$ are positive for all $x > a$.*

The above lemma, of course, holds if the words "nonnegative" and "positive" are replaced with "nonpositive" and "negative". We note that for $a \geqslant \epsilon$, the solution of (2.1) given by initial conditions

$$y(a) = y'(a) = y''(a) = 0, \qquad \text{and} \qquad [r(x) y''(x)]_{x=a} = 1$$

satisfies the hypotheses of Lemma 2.1. We call this solution the *principal solution of* (2.1) *at $x = a$*. The principal solution also satisfies the hypotheses of the following Lemma.

LEMMA 2.2. *Let $y(x)$ be a nontrivial solution of* (2.1) *and let $a \geqslant \epsilon$. If $y(a) \leqslant 0$, $y''(a) \leqslant 0$, $y'(a) \geqslant 0$, and $[r(x) y''(x)]'_{x=a} \geqslant 0$, then $y(x) < 0$, $y''(x) < 0$, $y'(x) > 0$, and $[r(x) y''(x)]' > 0$ for $x \in [\epsilon, a)$.*

We state next the following Lemma.

LEMMA 2.3. *Let a, b, and c be real numbers such that $\epsilon \leqslant a < b < c$, and let $y(x)$ be a solution of Eq.* (2.1). *If $y(a) = y(b) = y(c) = 0$, then $y'(b) \neq 0$.*

According to this lemma, if $y(x)$ is an oscillatory solution of (2.1) and $y(a) = y'(a) = 0$ for some $a > \epsilon$, then $y(x) \neq 0$ on $[\epsilon, a)$. Furthermore, by Lemma 2.1 we have $y''(a) [r(x) y''(x)]'_{x=a} < 0$.

Finally we have the following result.

LEMMA 2.4. *If $u(x)$ and $v(x)$ are essentially different solutions of* (2.1) *such that $u(a) = v(a) = u(b) = v(b) = 0$ for $\epsilon \leqslant a < b$, the zeros of $u(x)$ and $v(x)$ separate each other in (a, b).*

3. *General Properties*

We begin our discussion with the following result. For the case $r(x) \equiv 1$ the theorem below has already been stated by S. P. Hastings and A. C. Lazer [3].

THEOREM 3.1. *There exists a solution $w(x)$ of Eq.* (2.1) *which has the following properties:*

(i) $w(x)\, w'(x)\, w''(x)\, [r(x)\, w''(x)]' \neq 0$;

(ii) $\operatorname{sgn} w(x) = \operatorname{sgn} w''(x) \neq \operatorname{sgn} w'(x) = \operatorname{sgn}[r(x)\, w''(x)]'$;

(iii) $w(x)$ *and* $r(x)\, w''(x)$ *are asymptotic to a finite constant; and*

(iv) $\lim_{x\to\infty} w'(x) = \lim_{x\to\infty} w''(x) = \lim_{x\to\infty} [r(x)\, w''(x)]' = 0.$

Proof. For each positive integer $n \geqslant \epsilon$, let $y_n(x)$ be a solution of (2.1) such that $y_n(n) = y_n'(n) = y_n''(n) = 0$, and $[r(x)\, y_n''(x)]_{x=n}' < 0$. By Lemma 2.2, we have for each n,

$$y_n(x) > 0, \qquad y_n'(x) < 0, \qquad y_n''(x) > 0, \qquad \text{and} \qquad [r(x)\, y_n''(x)]' < 0 \tag{3.1}$$

for all $x \in [\epsilon, n)$.

Let $z_1(x)$, $z_2(x)$, $z_3(x)$, and $z_4(x)$ be four linearly independent solutions of (2.1). There then exist constants $c_{n_1}, c_{n_2}, c_{n_3}, c_{n_4}$, not all zero (since $y_n(x) \not\equiv 0$), such that

$$y_n(x) = c_{n_1} z_1(x) + c_{n_2} z_2(x) + c_{n_3} z_3(x) + c_{n_4} z_4(x) \tag{3.2}$$

and

$$c_{n_1}^2 + c_{n_2}^2 + c_{n_3}^2 + c_{n_4}^2 = 1. \tag{3.3}$$

By (3.3) the sequences $\{c_{ni}\}_{n=1}^{\infty}$ are bounded for each $i = 1, 2, 3, 4$. There thus exists a convergent subsequence $\{c_{n_k i}\}_{k=1}^{\infty}$ for each $i = 1, 2, 3, 4$. Let $\lim_{k\to\infty} c_{n_k i} = c_i$, $i = 1, 2, 3, 4$ and consider the solution

$$w(x) = c_1 z_1(x) + c_2 z_2(x) + c_3 z_3(x) + c_4 z_4(x). \tag{3.4}$$

By (3.3) we have $c_1^2 + c_2^2 + c_3^2 + c_4^2 = 1$, and hence $w(x) \not\equiv 0$. Furthermore, the sequences

$$\{y_{n_k}(x)\}_{k=1}^{\infty}, \qquad \{y_{n_k}'(x)\}_{k=1}^{\infty}, \qquad \{y_{n_k}''(x)\}_{k=1}^{\infty},$$

and

$$\{[r(x)\, y_{n_k}''(x)]'\}_{k=1}^{\infty}$$

converge uniformly to $w(x)$, $w'(x)$, $w''(x)$, and $[r(x)\, w''(x)]'$, respectively, on any finite subinterval of $[\epsilon, \infty)$.

By (3.1), we have $y_{n_k}(x) > 0$, $y_{n_k}'(x) < 0$, $y_{n_k}''(x) > 0$, and $[r(x)\, y_{n_k}''(x)]' < 0$ on $[\epsilon, n_k)$; accordingly we have

$$w(x) \geqslant 0, \qquad w'(x) \leqslant 0, \qquad w''(x) \geqslant 0, \qquad \text{and} \qquad [r(x)\, w''(x)]' \leqslant 0. \quad (3.5)$$

Suppose there exists a real number x_0 for which $w(x_0) = 0$. Since $w'(x) \leqslant 0$ and $w(x) \geqslant 0$, we must have $w(x) \equiv 0$ for all $x \geqslant x_0$, which contradicts the above statements.

Suppose $w'(x_0) = 0$ for some $x_0 \in (0, \infty)$. Then since $w'(x) \leqslant 0$ and $w''(x) \geqslant 0$, we have $w'(x) \equiv 0$ for all $x \geqslant x_0$. Hence, $w''(x) \equiv 0$, $r(x)\, w''(x) \equiv 0$, and $[r(x)\, w''(x)]' \equiv 0$ for all $x > x_0$. It follows that $p(x)\, w(x) = [r(x)\, w''(x)]'' \equiv 0$. This implies that $w(x) \equiv 0$, a contradiction.

We have shown that $w(x) > 0$ and $w'(x) < 0$. Similar proofs show that $w''(x) > 0$ and $[r(x)\, w''(x)]' < 0$. This completes the proof for parts (i) and (ii). Part (iii) follows immediately from (ii).

Since $w'(x) < 0$ and $w''(x) > 0$, we have $\lim_{x\to\infty} w'(x) \leqslant 0$. If $\lim_{x\to\infty} w'(x) < 0$, then $w(x)$ would eventually be negative, which contradicts $w(x) > 0$ for all x, as established above. Similar proofs hold for the other parts of (iv), and the proof of the theorem is complete.

We now turn our attention to the case when (2.1) has oscillatory solutions. One question which arises is whether or not an oscillatory solution $y(x)$ can have a multiple zero at some point $x = a \geqslant \epsilon$. Lemma 2.1 rules out the possibility of $y(x)$ having a triple zero at $x = a$. Lemma 2.3 essentially says that if $y(x)$ has a double zero at a point $x = a$, then $x = a$ is the first zero of $y(x)$ as x increases from ϵ, i.e., $y(x) \neq 0$ on the interval $[\epsilon, a)$.

The following theorem establishes the existence of oscillating solutions with a double zero when equation (2.1) is oscillatory. The proof of the theorem is essentially contained in the proof of Theorem 3.8 in Leighton and Nehari's paper [7], and for this reason the proof is omitted here. It should be pointed out, perhaps, that a proof may be made along the same lines as the proof of Theorem 3.1.

THEOREM 3.2. *If $v(x)$ is an oscillatory solution of* (2.1) *and if $v(a) = 0$ and $v(x)$ has a zero on $[\epsilon, a)$, then there exists an oscillatory solution $u(x)$ with the following properties:*

(i) $u(a) = u'(a) = 0$;

(ii) *the zeros of $u(x)$ and the conjugate points of $x = a$ separate each other;*

(iii) *$u(x)$ and $v(x)$ are essentially different.*

The question of the separation of zeros of two essentially different oscillatory solutions is a complex one. Simple examples show that their zeros may not separate each other. Indeed, the equation

$$y^{iv} - y = 0$$

has solutions

$$y_1(x) = \sin x$$

and

$$y_2(x) = \sin x - e^{-x}.$$

For x large enough $y_2(x)$ has two zeros between two consecutive zeros of $y_1(x)$, if $y_1(x) > 0$ between the consecutive zeros. Similarly $y_1(x)$ has two zeros between consecutive zeros of $y_2(x)$, if $y_2(x) < 0$ between the consecutive zeros. This type of separation plays a crucial role in studying the behavior of oscillatory solutions, and we shall return to this problem.

In order to investigate the separation of zeros of two essentially different oscillatory solutions, we require the following three lemmas.

LEMMA 3.1. *If $u(x)$ and $v(x)$ are essentially different oscillatory solutions of* (2.1) *and if $u(a) = v(a) = 0$ for some $a \geqslant \epsilon$, and if the zeros of $u(x)$ and $v(x)$ separate each other on (a, ∞), then $W[u, v]\,(x) \equiv u(x)\,v'(x) - v(x)\,u'(x)$ does not vanish on (a, ∞).*

Proof. Suppose there exists a number $b > a$ such that $W[u, v]\,(b) = 0$. There then exist constants c_1 and c_2, $c_1^2 + c_2^2 \neq 0$, such that

$$c_1u(b) + c_2v(b) = 0,$$
$$c_1u'(b) + c_2v'(b) = 0. \tag{3.6}$$

Let $y(x) = c_1u(x) + c_2v(x)$. Then by hypothesis and by (3.6), we have $y(a) = y(b) = y'(b) = 0$. Thus, by Lemma 2.3, $y(x) \neq 0$ for all $x > b$. We may assume $y(x) > 0$ for $x > b$, and so $c_1u(x) > -c_2v(x)$ for $x > b$. Note that if α and β are consecutive zeros of $v(x)$ for which $\operatorname{sgn} c_2 \neq \operatorname{sgn} v(x)$ for $x \in (\alpha, \beta)$, then $c_1u(x) > -c_2v(x) \geqslant 0$ on $[\alpha, \beta]$. Therefore, $u(x) \neq 0$ on $[\alpha, \beta]$, contrary to the hypothesis that the zeros of $u(x)$ and $v(x)$ separate each other. It follows that $W[u, v]\,(x) \neq 0$ for $x > a$.

LEMMA 3.2. *Suppose $u(x)$ and $v(x)$ are essentially different oscillatory solutions of* (2.1) *with $u(a) = v(a) = 0$ for some $a \geqslant \epsilon$. If the zeros of $u(x)$ and $v(x)$ separate each other, then every linear combination of $u(x)$ and $v(x)$ is oscillatory.*

Proof. Suppose $z(x) = c_1u(x) + c_2v(x) > 0$ for all $x \geqslant b$, for some $b > a$. Then $c_1u(x) > -c_2v(x)$ for $x \geqslant b$, and as in the proof of Lemma 3.1, there exists two consecutive zeros $\alpha < \beta$ of $v(x)$ such that $u(x) \neq 0$ on $[\alpha, \beta]$. A similar proof holds when $z(x)$ is eventually negative.

In order to simplify the statement of our theorems, we shall say that the zeros of two essentially different solutions $u(x)$ and $v(x)$ of (2.1) *separate pairwise* provided for some number b there is a pair of consecutive zeros of $u(x)$ greater than b followed by a pair of consecutive zeros of $v(x)$, and so on, and between the above pairs of consecutive zeros of $u(x)$ there is no zero of $v(x)$, and vice-versa. The functions $\sin x$ and $\sin x - e^{-x}$ in a previous example have this property.

LEMMA 3.3. *If $u(x)$ and $v(x)$ are oscillatory solutions of* (2.1) *such that $y(x) = u(x) - v(x)$ is nonoscillatory, the zeros of $u(x)$ and $v(x)$ separate pairwise.*

Proof. We may assume without loss of generality that $y(x) > 0$ for $x \geqslant b$ for some $b \geqslant \epsilon$. Then $u(x) > v(x)$ for $x \geqslant b$, and if $\alpha < \beta$ are consecutive zeros of $v(x)$, with $v(x) > 0$ on (α, β), then $u(x) > v(x) \geqslant 0$ on $[\alpha, \beta]$. If $\alpha < \beta$ are consecutive zeros of $u(x)$, with $u(x) < 0$ on (α, β), then $v(x) < u(x) \leqslant 0$ on $[\alpha, \beta]$.

To complete the proof we need only show that between any two consecutive zeros of $u(x)$ there is either no zero or two zeros of $v(x)$, and vice-versa. It is clear that we may limit ourselves to the proof for any two consecutive zeros of $u(x)$.

Suppose $\alpha < \beta$ are two consecutive zeros of $u(x)$. Then $v(\alpha) < u(\alpha) = 0$, and $v(\beta) < u(\beta) = 0$. Thus, $v(x)$ has an even number of zeros (possibly no zeros) between α and β. We appeal to ([7], Theorem 3.5), which states that the maximum number of zeros of $v(x)$ is three on the interval (α, β). Since $v(x)$ has an even number of zeros on (α, β), $v(x)$ has either no zero or two zeros on (α, β). The proof of Lemma 3.3 is complete.

COROLLARY 3.1. *If $u(x)$ and $v(x)$ are oscillatory solutions of* (2.1) *such that $y(x) = c_1u(x) + c_2v(x)$ is nonoscillatory, where c_1 and c_2 are constants, the zeros of $u(x)$ and $v(x)$ separate pairwise.*

Proof. Consider the solutions $u_1(x) = c_1u(x)$ and $v_1(x) = -c_2v(x)$. Note that $u_1(x)$ and $v_1(x)$ are oscillatory, and $y(x) = u_1(x) - v_1(x)$. By Lemma 3.3, the zeros of $u_1(x)$ and $v_1(x)$ separate pairwise. But the zeros of $u_1(x)$ and $v_1(x)$ are precisely the zeros of $u(x)$ and $v(x)$, respectively. Hence, the zeros of $u(x)$ and $v(x)$ separate pairwise.

The following theorem shows the existence of solutions whose zeros simply separate each other.

THEOREM 3.3. *If $u(x)$ and $v(x)$ are essentially different oscillatory solutions of* (2.1) *and if $u(a) = v(a) = 0$ for some $a \geqslant \epsilon$, then there exists an oscillatory solution $z(x)$ such that*

(i) $z(a) = 0$,

(ii) *the zeros of $u(x)$ and $z(x)$ separate each other on (a, ∞),*

(iii) *the zeros of $v(x)$ and $z(x)$ separate each other on (a, ∞).*

Proof. If the zeros of $u(x)$ and $v(x)$ separate each other, by Lemma 3.2, we may take $z(x) = u(x) + v(x)$. So we assume that the zeros of $u(x)$ and $v(x)$ do not separate each other.

Let $y_1(x)$, $y_2(x)$, $y_3(x)$ be three solutions of (2.1) which vanish at $x = a$ and which as three functions are linearly independent. Then any solution of (2.1) which vanishes at $x = a$ is a linear combination of $y_1(x)$, $y_2(x)$, and $y_3(x)$.

Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be the monotone sequences of zeros of $u(x)$ and $v(x)$, respectively, with $\alpha_1 > a$ and $\beta_1 > a$, $u(x) \neq 0$ on (a, α_1), and $v(x) \neq 0$ on (a, β_1). For each positive integer n, let $z_n(x)$ be the essentially unique solution which vanishes at $x = a, \alpha_n, \beta_n$. There then exist constants c_{1n}, c_{2n}, c_{3n} such that

$$z_n(x) = c_{1n} y_1(x) + c_{2n} y_2(x) + c_{3n} y_3(x), \tag{3.7}$$

$$c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1. \tag{3.8}$$

Since the sequences $\{c_{1n}\}_{n=1}^{\infty}$, $\{c_{2n}\}_{n=1}^{\infty}$ and $\{c_{3n}\}_{n=1}^{\infty}$ are bounded, each possesses convergent subsequences, say $\{c_{1n_j}\}_{j=1}^{\infty}$, $\{c_{2n_j}\}_{j=1}^{\infty}$, $\{c_{3n_j}\}_{j=1}^{\infty}$, respectively. Let $\lim_{j\to\infty} c_{in_j} = c_i$ for $i = 1, 2, 3$, and define

$$z(x) \equiv c_1 y_1(x) + c_2 y_2(x) + c_3 y(x).$$

By (3.8) it is clear that $c_1{}^2 + c_2{}^2 + c_3{}^2 = 1$, and we have $z(x) \not\equiv 0$ and $z(a) = 0$.

Now let $n \geqslant 1$ and consider the consecutive zeros α_n and α_{n+1} of $u(x)$. Note that both $z_m(x)$ and $u(x)$ vanish at $x = a$ and at $x = \alpha_m$. Consequently, by Lemma 2.4, the zeros of $z_m(x)$ and $u(x)$ separate in the interval (a, α_m). It follows that, for $m > n + 1$, $z_m(x)$ has a zero in the interval (α_n, α_{n+1}). Thus $z(x)$ has a zero in $[\alpha_n, \alpha_{n+1}]$ and hence is oscillatory. A similar proof shows the existence of a zero of $z(x)$ in the interval $[\beta_n, \beta_{n+1}]$.

Now if $z(x)$ had three zeros in common with $u(x)$, $u(x)$ and $z(x)$ would be constant multiples of each other, by a theorem due to Leighton and Nehari ([7] p. 239). This would say that $u(x)$ had a zero in $[\beta_n, \beta_{n+1}]$ for $n \geqslant 1$, which contradicts either the fact that $u(x)$ and $v(x)$ are essentially different or the assumption that their zeros do not separate each other. Therefore, $z(x)$ has a zero in (α_n, α_{n+1}), at least for n large enough.

Again, similar analysis shows that $z(x)$ has a zero in (β_n, β_{n+1}), for n large enough.

We now show that $z(x)$ has exactly one zero in (α_n, α_{n+1}). Suppose $z(x)$ had two zeros in (α_n, α_{n+1}), say $\delta_1 < \delta_2$. Then, by Lemma 1.1, there exists a constant k such that $y(x) = z(x) - ku(x)$ has a double zero at $x = \gamma \in (\delta_1, \delta_2) \subset (\alpha_n, \alpha_{n+1})$. If $y''(\gamma)\,[r(x)\,y''(x)]'_{x=\gamma} < 0$, by Lemma 2.2 $y(x) \neq 0$ on (ϵ, γ), which contradicts $y(a) = 0$. Hence $y''(\gamma)\,[r(x)\,y''(x)]'_{x=\gamma} \geqslant 0$. By Lemma 2.1, $y(x)$ is nonoscillatory. Applying Corollary 3.1 we have that the zeros of $u(x)$ and $z(x)$ separate pairwise. But this means that there exists two consecutive zeros of $u(x)$ such that $z(x) \neq 0$ between them. From this contradiction we infer that $z(x)$ has exactly one zero in (α_n, α_{n+1}) for $n \geqslant 1$.

A similar proof will show that $z(x)$ has exactly one zero in (β_n, β_{n+1}). Therefore, the zeros of $z(x)$ separate the zeros of $u(x)$ and separate the zeros of $v(x)$.

The following corollary is an immediate consequence of Theorem 3.3 and Lemma 3.2.

COROLLARY 3.2. *Let $u(x)$, $v(x)$, and $z(x)$ be solutions of* (2.1) *as given in Theorem 3.3. Then every linear combination of $u(x)$ and $z(x)$, or of $v(x)$ and $z(x)$, is oscillatory.*

The next result provides another method of generating oscillatory solutions. The reader should note the role played by the nonoscillatory solution $w(x)$ in this theorem.

THEOREM 3.4. *Let $w(x)$ be a nonoscillatory solution of* (2.1) *such that*

$$\operatorname{sgn} w(x) = \operatorname{sgn} w''(x) \neq \operatorname{sgn} w'(x) = \operatorname{sgn}[r(x)\,w''(x)]'. \tag{3.9}$$

Let $z(x)$ be any oscillatory solution of (2.1). *Then, for each $a \in [\epsilon, \infty)$ for which $z(a) \neq 0$, there exists a constant $k(a)$ such that the solution*

$$y(x) = z(x) - k(a)\,w(x)$$

is oscillatory and $y(a) = 0$.

Proof. Let $w(x)$ be a nonoscillatory solution of (2.1) satisfying condition (3.9). Suppose $w(x) > 0$. Let $z(x)$ be oscillatory and $z(a) > 0$. [These assumptions on the signs of $z(a)$ and $w(a)$ are used only to fix the ideas.] Define $k(a)$ to be the solution of the equation

$$z(a) - k(a)\,w(a) = 0$$

and define $y(x) = z(x) - k(a)\,w(x)$. Note that $y(a) = 0$.

Since $z(a) > 0$ and $w(a) > 0$, we have $k(a) > 0$. Suppose $y(x)$ is not oscillatory. If $y(x) \equiv 0$, then $z(x) \equiv k(a)\,w(x)$ which cannot happen since

$z(x)$ is oscillatory and $w(x)$ is nonoscillatory. Suppose then that $y(x)$ is nonoscillatory. There then exists a number $b > a$ such that

$$y(x)\, y'(x)\, y''(x)\, [r(x)\, y''(x)]' \neq 0$$

for all $x \geqslant b$.

If $y(x) \geqslant 0$, then $z(x) \geqslant k(a)\, w(x) > 0$ for all $x \geqslant b$, contradicting the oscillatory character of $z(x)$. So $y(x) < 0$. If $y'(x) \leqslant 0$, then $z'(x) \leqslant k(a)\, w'(x) < 0$ for $x \geqslant b$, again a contradiction. Thus, $y'(x) > 0$. Similar arguments show $y''(x) < 0$ and $[r(x)\, y''(x)]' > 0$ for all $x \geqslant b$. But by Lemma 2.2, $y(x) \neq 0$ for $x \in (\epsilon, b)$, contradicting $y(a) = 0$. Hence $y(x)$ is oscillatory. The proof is complete.

As a note, $k(a) \equiv z(a)/w(a)$ for each a for which $z(a) \neq 0$. Since $w(a) \neq 0$, $k(a)$ is a continuous function of a with removable discontinuities at the zeros of $z(x)$.

Let $w(x)$ and $z(x)$ be solutions of (2.1) as in Theorem 3.4. The following corollary is then of interest in that it shows that a certain intimate relationship exists between $\lim_{x\to\infty} w(x)$ and the existence of $\lim_{x\to\infty} z(x)$.

COROLLARY 3.3. *If* $w(x) \to c \neq 0$, *then for any oscillatory solution* $z(x)$, *we have that* $\lim_{x\to\infty} z(x)$ *does not exist.*

Proof. Suppose $\lim_{x\to\infty} z(x)$ did exist. Then, since $z(x)$ is oscillatory, we have $\lim_{x\to\infty} z(x) = 0$. Let $a \in [\epsilon, \infty)$ with $z(a) \neq 0$, and define $k(a)$ as in Theorem 3.4. By Theorem 3.1, $\lim_{x\to\infty} w(x)$ exists, and, say, $\lim_{x\to\infty} w(x) = c$. Consider $y(x) = z(x) - k(a)\, w(x)$. Since $\lim_{x\to\infty} z(x)$ and $\lim_{x\to\infty} w(x)$ both exist and are finite, we have that $\lim_{x\to\infty} y(x)$ exists. Since by Theorem 3.4, $y(x)$ is oscillatory, $\lim_{x\to\infty} y(x) = 0$, and

$$w(x) = \frac{1}{k(a)}\, [z(x) - y(x)].$$

So

$$0 \neq c = \lim_{x\to\infty} w(x) = \lim_{x\to\infty} \frac{1}{k(a)}\, [z(x) - y(x)] = 0,$$

a contradiction. It follows that $\lim_{x\to\infty} z(x)$ does not exist.

Note that Theorem 3.4 and Corollary 3.3 hold for any solution $w(x)$ of (2.1) which satisfies the conditions

$$w(x)\, w'(x)\, w''(x)\, [r(x)\, w''(x)]' \neq 0 \tag{3.10}$$

and

$$\operatorname{sgn} w(x) = \operatorname{sgn} w''(x) \neq \operatorname{sgn} w'(x) = \operatorname{sgn}[r(x)\, w''(x)]'. \tag{3.11}$$

Recall such a solution exists by Theorem 3.1.

THEOREM 3.5. *Let* $w(x)$ *satisfy conditions* (3.10) *and* (3.11), *and suppose* $\liminf_{x\to\infty} p(x) \neq 0$. *Then* $\lim_{x\to\infty} w(x) = 0$.

Proof. Since

$$\operatorname{sgn}[r(x)\, w''(x)]' \neq \operatorname{sgn}[r(x)\, w''(x)] = \operatorname{sgn}[p(x)\, w(x)] = \operatorname{sgn}[r(x)\, w''(x)]'',$$

we have that $\lim_{x\to\infty} r(x)\, w''(x)$ exists, and

$$\lim_{x\to\infty} [r(x)\, w''(x)]' = \lim_{x\to\infty} [r(x)\, w''(x)]'' = 0.$$

So

$$0 = \lim_{x\to\infty} [r(x)\, w''(x)]'' = \lim_{x\to\infty} p(x)\, w(x).$$

By (3.11) $\lim_{x\to\infty} w(x)$ clearly exists. Hence, since $\liminf_{x\to\infty} p(x) \neq 0$, we must have $\lim_{x\to\infty} w(x) = 0$.

Interestingly enough the hypothesis in Theorem 3.5 is independent of any conditions on $r(x)$, except, of course, the standing hypotheses. For the case $r(x) \equiv 1$, Hastings and Lazer [3] have shown that if $p \in C'$, $p'(x) \geqslant 0$, and $\lim_{x\to\infty} p(x) = +\infty$, then all oscillatory solutions tend to zero, as $x \to \infty$.

We conclude this section with the following theorem concerning the behavior of nonoscillatory solutions.

THEOREM 3.6. *Any nonoscillatory solution* $y(x)$ *of* (2.1) *satisfies one of the following conditions for all* $x \geqslant a$ *for some* $a \geqslant \epsilon$

(i) $\operatorname{sgn} y(x) = \operatorname{sgn} y'(x) = \operatorname{sgn} y''(x) = \operatorname{sgn}[r(x)\, y''(x)]'$,

(ii) $\operatorname{sgn} y(x) = \operatorname{sgn} y''(x) \neq \operatorname{sgn} y'(x) = \operatorname{sgn}[r(x)\, y''(x)]'$,

(iii) $\operatorname{sgn} y(x) = \operatorname{sgn} y'(x) \neq \operatorname{sgn} y''(x) = \operatorname{sgn}[r(x)\, y''(x)]'$,

(iv) $\operatorname{sgn} y(x) = \operatorname{sgn} y'(x) = \operatorname{sgn} y''(x) \neq \operatorname{sgn}[r(x)\, y''(x)]'$,

(v) $\operatorname{sgn} y(x) = \operatorname{sgn} y''(x) = \operatorname{sgn}[r(x)\, y''(x)]' \neq \operatorname{sgn} y'(x)$.

Proof. The only other possible conditions are

$$\operatorname{sgn} y(x) = \operatorname{sgn} y'(x) \neq \operatorname{sgn} y''(x) \neq \operatorname{sgn}[r(x)\, y''(x)]', \tag{3.12}$$

$$\operatorname{sgn} y(x) \neq \operatorname{sgn} y'(x) = \operatorname{sgn} y''(x) \neq \operatorname{sgn}[r(x)\, y''(x)]', \tag{3.13}$$

$$\operatorname{sgn} y(x) \neq \operatorname{sgn} y'(x) = \operatorname{sgn} y''(x) = \operatorname{sgn}[r(x)\, y''(x)]'. \tag{3.14}$$

Suppose (3.12) holds and assume that $y(x) > 0$ eventually. Then, $[r(x)\, y''(x)]' > 0$, and $[r(x)\, y''(x)]'' = p(x)\, y(x) > 0$. Hence eventually $r(x)\, y''(x) > 0$—which contradicts $y''(x) < 0$ for x sufficiently large.

Suppose (3.13) or (3.14) holds, and assume $y(x) > 0$. Then $y'(x) < 0$ and $y''(x) < 0$, which implies that $y(x) < 0$ eventually—a contradiction.

Accordingly, conditions (i)-(v) are the only possibilities.

Before we leave this subject, let us point out that subsequent to the standing hypotheses on $r(x)$ and $p(x)$, Eq. (2.1) always has solutions which satisfy conditions (i) and (ii) of Theorem 3.6. (See Lemma 2.1 and Theorem 3.1.)

4. *Bounded Oscillatory Solutions*

In this section we restrict our attention to the case when Eq. (2.1) is not only oscillatory, but also each oscillatory solution is bounded. Among other things, we are able to obtain a representation for an oscillatory solution. Considering the complicated behavior of solutions of (2.1), this representation is remarkably simple.

First we give two conditions which will guarantee the existence of bounded oscillatory solutions.

THEOREM 4.1. *If $r'(x) \geqslant 0$, $p'(x) \leqslant 0$, and $p(x) \nrightarrow 0$, r, $p \in C'$, all oscillatory solutions are bounded.*

Proof. Consider the following identity, which may be verified by differentiation:

$$\begin{aligned} G[y(x)] &\equiv r(x)\, y''^2(x) - 2y'(x)\,[r(x)\, y''(x)]' + p(x)\, y^2(x) \\ &= G[y(a)] - \int_a^x [r'(t)\, y''^2(t) - p'(t)\, y^2(t)]\, dt, \end{aligned}$$

where $y(x)$ is a solution of (2.1), and $a \geqslant \epsilon$. Let $\{b_i\}_{i=1}^{\infty}$ be the monotone increasing sequence of zeros of $y'(x)$ greater than a. Under the conditions of the theorem, $G[y(x)]$ is nonincreasing in x. Thus, we have

$$\begin{aligned} p(b_i)\, y^2(b_i) &\leqslant r(b_i)\, y''^2(b_i) + p(b_i)\, y^2(b_i) \\ &= G[y(b_i)] \leqslant G[y(a)], \end{aligned}$$

and, accordingly,

$$y^2(b_i) \leqslant \frac{G[y(a)]}{p(b_i)}.$$

Since $p(x) \nrightarrow 0$ as $x \to \infty$, $1/p(b_i)$ is bounded, and it follows that $|\, y(x)|$ is bounded.

Hastings and Lazer [3] have shown that if $r(x) \equiv 1$ and $p'(x) \geqslant 0$, the oscillatory solutions are bounded. In view of the previous lemma the equation $y^{(iv)} - p(x)\, y = 0$, where $p'(x)$ doesn't change sign, will have bounded oscillatory solutions if $\lim_{x\to\infty} p(x) \neq 0$.

In considering the oscillatory character of the second-order equation

$$[r(x)\, y']' + p(x)\, y = 0, \tag{4.1}$$

where $r(x)$ and $p(x)$ are positive and continuous, the function $[r(x)\,p(x)]'$ plays a fundamental role. The following theorem therefore is of interest since it contains a condition on the analogue of the function $[r(x)\,p(x)]'$.

THEOREM 4.2. *If the functions* $[r(x)\,p^3(x)]'$ *and* $[r(x)\,(1/p(x))'']'$ *are continuous and nonnegative for* $x \geqslant a \geqslant \epsilon$, *then oscillatory solutions are bounded.*

Proof. Consider the following identity which may be verified by differentiation:

$$\begin{aligned} H[y(x)] &\equiv \frac{2y'(x)\,[r(x)\,y''(x)]'}{p(x)} - y^2(x) - \frac{r(x)\,y''^2(x)}{p(x)} \\ &\quad + \frac{2p'(x)\,r(x)\,y''(x)\,y'(x)}{p^2(x)} + r(x)\left[\frac{1}{p(x)}\right]'' y'^2(x) \\ &= H[y(a)] + \int_a^x \left[\frac{(r(t)\,p^3(t))'}{p^4(t)}\,y''^2(t) + \left(r(t)\left[\frac{1}{p(x)}\right]''\right)' y'^2(t)\right] dt. \end{aligned}$$

Under the conditions of the theorem, $H[y(x)]$ is nondecreasing. So if $y(x)$ is oscillatory, and $\{b_i\}_{i=1}^{\infty}$ is the increasing sequence of points where $y(x)$ has relative maxima or minima, then $y'(b_i) = 0$. Furthermore

$$-y^2(b_i) \geqslant -y^2(b_i) - \frac{r(b_i)\,y''^2(b_i)}{p(b_i)} = H[y(b_i)] \geqslant H[y(a)].$$

Hence, $y^2(b_i) \leqslant -H[y(a)]$. So $|y(x)|$ is bounded, and the proof is complete.

We now state the following theorem.

THEOREM 4.3. *Suppose* $y(x)$ *is a nonoscillatory solution of* (2.1), *and suppose* $y(x)$ *is bounded. If* $r'(x) \leqslant 0$ *for large* x, *then* $y(x)$ *satisfies*

(i) $y(x)\,y'(x)\,y''(x)\,[r(x)\,y''(x)]' \neq 0$ *for* $x \in [\epsilon, \infty)$

and

(ii) $\operatorname{sgn} y(x) = \operatorname{sgn} y''(x) \neq \operatorname{sgn} y'(x) = \operatorname{sgn}[r(x)\,y''(x)]'$.

Proof. Since $y(x)$ is nonoscillatory, there exists a number $b > \epsilon$ such that $y(x)\,y'(x)\,y''(x)\,[r(x)\,y''(x)]' \neq 0$ for $x \geqslant b$. We show that for $x \geqslant b$ each of the following inequalities holds:

(1) $y''(x)\,[r(x)\,y''(x)]' < 0$,

(2) $y(x)\,[r(x)\,y''(x)]' < 0$,

(3) $y'(x)\,y''(x) < 0$.

Suppose (1) does not hold. Then,

$$0 \leqslant y''(x)\,[r(x)\,y''(x)]' = r'(x)\,y''^2(x) + r(x)\,y''(x)\,y'''(x).$$

Since $r'(x) \leqslant 0$, we have $y''(x)\,y'''(x) \geqslant 0$, which implies $y'(x)\,y''(x) \geqslant 0$, which in turn implies $y(x)\,y'(x) \geqslant 0$. Then the last two inequalities imply $|\,y(x)|$ is unbounded, contradicting the hypothesis that $y(x)$ is bounded.

Suppose (2) does not hold. Then

$$0 \leqslant y(x)\,[r(x)\,y''(x)]' = \frac{p(x)\,y(x)\,[r(x)\,y''(x)]'}{p(x)}$$
$$= \frac{[r(x)\,y''(x)]''\,[r(x)\,y''(x)]'}{p(x)}.$$

Since $p(x) > 0$, we have $[r(x)\,y''(x)]''\,[r(x)\,y''(x)]' \geqslant 0$ which implies $[r(x)\,y''(x)]\,[r(x)\,y''(x)]' \geqslant 0$, or $y''(x)\,[r(x)\,y''(x)]' \geqslant 0$. The above discussion shows that this last inequality leads to a contradiction.

Suppose (3) does not hold. Then

$$y'(x)\,y''(x) \geqslant 0$$

for large x. By the above discussion this last inequality also leads to a contradiction.

Thus, we have shown that (i) and (ii) hold for $x \geqslant b$. By Lemma 2.2, (i) and (ii) hold for $x \in [\epsilon, b)$, and the proof is complete.

The following corollary is immediate.

COROLLARY 4.1. *If* $r'(x) \leqslant 0$, *any solution which vanishes on* (ϵ, ∞) *is either unbounded or oscillatory.*

For the remaining theorems in this portion of the paper, we assume that oscillatory solutions are bounded.

THEOREM 4.4. *If* $u(x)$ *and* $v(x)$ *are essentially different oscillatory solutions of* (2.1) *such that* $u(a) = v(a) = 0$ *for some* $a \geqslant \epsilon$, *then every linear combination of* $u(x)$ *and* $v(x)$ *is oscillatory, or identically zero.*

Proof. Let $y(x)$ be the principal solution of (2.1) at $x = a$. First we show that $u(x)$, $v(x)$, and $y(x)$ are linearly independent. Consider constants c_1, c_2, c_3 such that

$$c_1u(x) + c_2v(x) + c_3y(x) \equiv 0.$$

Note that if $c_3 = 0$, we contradict the hypothesis that $u(x)$ and $v(x)$ are essentially different solutions, unless $c_1 = c_2 = 0$ also.

But then if either c_1 or c_2 are zero and $c_3 \neq 0$, then $y(x)$ and either $u(x)$ or $v(x)$ would be constant multiples of each other, which is also a contradiction. Accordingly, $c_1c_2 \neq 0$. Then,

$$c_1u(x) \equiv -\,c_2v(x) - c_3y(x).$$

Recall that $y(x)$ is unbounded; so at the zeros of $v(x)$, $u(x)$ would be unbounded—a contradiction. It follows that $c_1 = c_2 = c_3 = 0$, and thus that $u(x)$, $v(x)$, and $y(x)$ are three solutions of (2.1) which vanish at $x = a$ and which, as functions, are linearly independent.

We return to the proof of Theorem 3.3, and take $y_1(x) \equiv u(x)$, $y_2(x) \equiv v(x)$, $y_3(x) \equiv y(x)$. Then the solution $z(x)$ of Theorem 3.3 has the following properties:

$$z(x) \equiv c_1u(x) + c_2v(x) + c_3y(x), \tag{4.2}$$

$$c_1^2 + c_2^2 + c_3^2 = 1, \tag{4.3}$$

$$z(a) = 0, \textit{ and } z(x) \textit{ is oscillatory}, \tag{4.4}$$

$$\textit{the zeros of } u(x) \textit{ and } z(x) \textit{ separate each other}, \tag{4.5}$$

and

$$\textit{the zeros of } v(x) \textit{ and } z(x) \textit{ separate each other}. \tag{4.6}$$

By Corollary 3.2 the solution $z(x) - c_1u(x)$ is oscillatory and hence bounded. But

$$z(x) - c_1u(x) = c_2v(x) + c_3y(x). \tag{4.7}$$

The right-hand side of (3.7) is unbounded, unless $c_3 = 0$. It follows that $c_3 = 0$, and $z(x) = c_1u(x) + c_2v(x)$.

Now let $z_1(x) = k_1u(x) + k_2v(x)$ for constants k_1 and k_2, not both zero. By (4.3), $c_1^2 + c_2^2 = 1$. Assume without loss of generality that $c_2 \neq 0$. Then

$$v(x) = \frac{1}{c_2}\,(z(x) - c_1u(x)).$$

Hence,

$$z_1(x) = k_1u(x) + k_2\left[\frac{1}{c_2}\,(z(x) - c_1u(x))\right] = \left(k_1 - \frac{k_2c_1}{c_2}\right)u(x) + \frac{k_2}{c_2}\,z(x)$$

which is oscillatory, by Corollary 3.2. Therefore, every linear combination of $u(x)$ and $v(x)$ is oscillatory.

The following theorem gives a representation for oscillatory solutions vanishing at $x = a$.

THEOREM 4.5. *Suppose $u(x)$ and $v(x)$ are essentially different oscillatory solutions of* (2.1) *such that $u(a) = v(a) = 0$ for some $a \in [\epsilon, \infty)$, and suppose that $u'(a) = 0$, $u''(a) = 1$, and $v'(a) = 1$. Then, every oscillatory solution of* (2.1) *vanishing at $x = a$ is a linear combination of $u(x)$ and $v(x)$.*

Proof. Let $z(x)$ be an oscillatory solution such that $z(a) = 0$, and let

$z'(a) = k_1$, $z''(a) = k_2$. By Lemma 2.1, $k_1^2 + k_2^2 \neq 0$. Consider the solutions $y(x)$ given by

$$y(x) \equiv z(x) - [k_1 v(x) + (k_2 - k_1 v''(a))\, u(x)]. \tag{4.8}$$

By Theorem 4.4, $y(x)$ is either oscillatory or identically zero.

We show that $y(x)$ is not oscillatory. By (4.8) it is clear that $y(a) = 0$. We calculate

$$\begin{aligned} y'(a) &= z'(a) - [k_1 v'(a) + (k_2 - k_1 v''(a))\, u'(a)] \\ &= k_1 - k_1 = 0 \\ y''(a) &= z''(a) - [k_1 v''(a) + (k_2 - k_1 v''(a))\, u''(a)] \\ &= k_2 - k_1 v''(a) - k_2 + k_1 v''(a) = 0. \end{aligned}$$

So, by Lemma 2.1, $y(x)$ is not oscillatory and

$$z(x) \equiv k_1 v(x) + (k_2 - k_1 v''(a))\, u(x). \tag{4.9}$$

The existence of an oscillatory solution having the properties of $u(x)$ and $v(x)$ in the above theorem was established in Theorem 3.2. The remarkable thing about Theorem 4.5 is that the function $[r(x)\, z''(x)]'$ plays no explicit part in the representation (4.9). So given an arbitrary solution $z(x)$ with

$$z(a) = 0, \qquad [z'(a)]^2 + [z''(a)]^2 \neq 0,$$

one can choose $[r(x)\, z''(x)]'_{x=a}$ in such a way that $z(x)$ will be oscillatory; namely,

$$[r(x)\, z''(x)]'_{x=a} = k_1 [r(x)\, v''(x)]'_{x=a} + [k_2 - k_1 v''(a)]\, [r(x)\, u''(x)]'_{x=a}\,.$$

THEOREM 4.6. *If $u(x)$ and $v(x)$ are oscillatory solutions of* (2.1) *that have two zeros* (*coincident, or distinct*) *in common, they are constant multiples of each other.*

Proof. Let $v(x)$ and $u(x)$ be solutions of (2.1) with the properties $v(a) = 0$, $v'(a) = 1$, $u(a) = u'(a) = 0$, and $u''(a) = 1$. Then, if $z(x)$ is an oscillatory solution such that $z(a) = z'(a) = 0$, $z''(a) = k_2 \neq 0$, then by (4.9), $z(x) = k_2 u(x)$.

Suppose now that $z_1(x)$ and $z_2(x)$ are essentially different oscillatory solutions of (2.1) with $z_1(a) = z_2(a) = z_1(b) = z_2(b) = 0$ for $a < b$. Note $z_1'(b)\, z_2'(b) \neq 0$ by Lemma 2.3. Consider the solution of (2.1) given by

$$y(x) \equiv z_2'(b)\, z_1(x) - z_1'(b)\, z_2(x). \tag{4.10}$$

Note that $y(a) = y(b) = y'(b) = 0$ from (4.10). If $y(x) \not\equiv 0$, then by Lemma 2.3, $y(x) \neq 0$ on (b, ∞). So $y(x)$ is identically zero, by Theorem 4.4. Hence, $z_1'(b)\, z_2(x) \equiv z_2'(b)\, z_2(x)$, for all x. This contradicts the assumption that $z_1(x)$ and $z_2(x)$ are essentially different. So no such $z_1(x)$ and $z_2(x)$ exist, and the proof is complete.

The above theorem shows that in Lemma 2.4 the hypotheses are vacuous if $u(x)$ and $v(x)$ in that lemma are known to be oscillatory.

We conclude this section of the paper with the following result. This theorem provides quite a remarkable representation for *any* oscillatory solution of (2.1).

THEOREM 4.7. *If $z(x)$ is an oscillatory solution of* (2.1) *and $z(a) \neq 0$, there then exists constants c_1, c_2, $k(a)$ such that*

$$z(x) \equiv k(a)\, w(x) + c_1 u(x) + c_2 v(x),$$

where $u(x)$ and $v(x)$ have the properties

$$u(a) = u'(a) = v(a) = 0, \qquad u''(a) = v'(a) = 1,$$

and $w(x)$ has the properties (3.10) *and* (3.11).

Proof. The existence of $k(a)$ follows immediately from Theorem 3.4. Since the function $z(x) - k(a)\, w(x)$ is also oscillatory by Theorem 3.4, we have, by Theorem 4.5,

$$z(x) - k(a)\, w(x) \equiv c_1 u(x) + c_2 v(x)$$

for some constants c_1 and c_2. The proof is complete.

PART II

5. *Properties of Oscillatory Solutions of* (1.4)

We now turn our attention to the equation

$$[r(x)\, y'']'' + p(x)\, y = 0, \tag{5.1}$$

where $r(x)$ and $p(x)$ are positive and continuous on an interval $[\epsilon, \infty)$ for some $\epsilon > 0$.

In the study of Eq. (5.1) the following identity plays an important role:

$$\begin{aligned} F[y(x)] &\equiv r(x)\, y'(x)\, y''(x) - y(x)\, [r(x)\, y''(x)]' \\ &= F[y(a)] + \int_a^x [r(t)\, y''^2(t) + p(t)\, y^2(t)]\, dt, \end{aligned}$$

where $y(x)$ is a solution of Eq. (5.1) and $a \geqslant \epsilon$. Since $r(x) > 0$ and $p(x) > 0$, $F[y(x)]$ is strictly increasing in the variable x for every nontrivial solution $y(x)$. It is important to note that if $y(x)$ is a nontrivial solution with a double zero at $x = b \geqslant \epsilon$, then $F[y(b)] = 0$. Consequently, $y(x)$ can have at most one double zero.

In this section we shall be concerned only with the case when Eq. (5.1) is known to be oscillatory. Leighton and Nehari [7] have shown that if Eq. (5.1) has an oscillatory solution, then every solution of Eq. (5.1) is oscillatory.

Criteria for determining whether or not Eq. (5.1) is oscillatory are known. For example, for equation

$$y^{iv} + p(x)\,y = 0, \qquad p(x) > 0 \tag{5.2}$$

and $p(x)$ continuous, Leighton and Nehari have shown that $\int_a^\infty x^2 p(x)\,dx = \infty$ is sufficient for Eq. (5.2) to be oscillatory. One may obtain a criterion for the equation

$$[r(x)\,y'']'' + y = 0, \tag{5.3}$$

where $r(x) > 0$ and $r(x)$ continuous, by noting that if $y(x)$ is a solution of (5.3), then $z(x) = r(x)\,y''(x)$ is a solution of the equation

$$z^{iv} + \frac{z}{r(x)} = 0. \tag{5.4}$$

Applying the above result to (5.4) and noting that $z(x)$ is oscillatory if and only if $y(x)$ is oscillatory, one obtains $\int_a^\infty (x^2/r(x))dx = \infty$ as a sufficient condition for Eq. (5.3) to be oscillatory.

Although the main objective in this section is not to give conditions under which Eq. (5.1) will be oscillatory, we shall provide a rather simple condition for oscillation. The following theorem is a special case of a more complicated result of Leighton and Nehari [7]. We include the result because of the rather simple hypotheses, and we give a different proof.

THEOREM 5.1. *If $r'(x) \leqslant 0$ and $p'(x) \geqslant 0$ for x sufficiently large, then Eq. (5.1) is oscillatory.*

Proof. Suppose $y(x)$ is a nonoscillatory solution of Eq. (5.1). Then there exists $a \geqslant \epsilon$ such that $y(x)\,y'(x)\,y''(x)\,[r(x)\,y''(x)]' \neq 0$ on the interval $[a, \infty)$. We assume then $y(x) > 0$ on $[a, \infty)$.

First suppose that $y'(x) > 0$ on $[a, \infty)$. Then for x sufficiently large,

$$[r(x)\,y''(x)]'' = -\,p(x)\,y(x) < 0, \tag{5.5}$$

and

$$[r(x)\,y''(x)]''' = -\,p(x)\,y'(x) - p'(x)\,y(x) < 0. \tag{5.6}$$

Now, (5.5) and (5.6) imply

$$[r(x)\,y''(x)]' < 0 \qquad \text{on} \qquad [a, \infty). \tag{5.7}$$

By (5.6) and (5.7), we have $r(x)\,y''(x) < 0$; that is, $y''(x) < 0$ on $[a, \infty)$. Since $r'(x) \leqslant 0$ and

$$0 > [r(x)\,y''(x)]' = r'(x)\,y''(x) + r(x)\,y'''(x),$$

we have $y'''(x) < 0$ on $[a, \infty)$. Therefore, $y'(x) < 0$ on $[a, \infty)$—which contradicts our assumption that $y'(x) > 0$ on $[a, \infty)$.

Suppose then $y'(x) < 0$ on $[a, \infty)$. Since $y(x) > 0$ on $[a, \infty)$, we must have $y''(x) > 0$ on $[a, \infty)$. Consequently, $[r(x)\,y''(x)]' < 0$ on $[a, \infty)$, and hence $[r(x)\,y''(x)]'' > 0$ on $[a, \infty)$. But $0 < [r(x)\,y''(x)]'' = -p(x)\,y(x) < 0$ —a contradiction. Since both the assumptions $y'(x) > 0$ and $y'(x) < 0$ on $[a, \infty)$ lead to contradictions, $y(x)$ must be oscillatory.

We next consider the behavior of oscillatory solutions of Eq. (5.1). Many of the following theorems were given by Svec [8] for the special case $r(x) \equiv 1$. One should not expect all of Svec's results to carry over when $r(x)$ is not necessarily $\equiv 1$, in particular, the boundedness of solutions. However a surprising number of Svec's results hold when $r(x)$ is a positive and continuous function.

THEOREM 5.2. *Let $y(x)$ be a nontrivial solution of Eq.* (5.1) *such that for some $a \geqslant \epsilon$, $y(a) = 0$ and either*

(i) $y'(a)\,y''(a) \geqslant 0$,

or

(ii) $y''(a)\,[r(x)\,y''(x)]'_{x=a} \geqslant 0$.

If $b > a$ and $y(b) = 0$, then

$$y'(b)\,y''(b)\,[r(x)\,y''(x)]'_{x=b} \neq 0, \tag{5.8}$$

and

$$\operatorname{sgn} y'(b) = \operatorname{sgn} y''(b) = \operatorname{sgn}[r(x)\,y''(x)]'_{x=b}\,. \tag{5.9}$$

Proof. First suppose (i) holds. Since $F[y(x)]$ is increasing,

$$0 \leqslant r(a)\,y'(a)\,y''(a) = F[y(a)] < F[y(b)] = r(b)\,y'(b)\,y''(b).$$

Therefore, $y'(b)\,y''(b) > 0$. Since $y(x) \not\equiv 0$, there exists a point $x = r_1$ such that $y'(r_1) = 0$, $y'(x) \neq 0$ on the interval (r_1, b) and $a < r_1 < b$. Since $y(b) = 0$,

$$y(x)\,y'(x) < 0, \qquad (r_1 < x < b), \tag{5.10}$$

and

$$0 \leqslant F(y(a)] < F[y(r_1)] = -y(r_1)\,[r(x)\,y''(x)]'_{x=r_1}\,. \tag{5.11}$$

Furthermore, $[r(x)\,y''(x)]'' = -p(x)\,y(x)$ for $x \geqslant \epsilon$. Consequently, $\operatorname{sgn}[r(x)\,y''(x)]'' \neq \operatorname{sgn} y(x)$ for $x \geqslant \epsilon$, and so, by (5.11), $[r(x)\,y''(x)]' \neq 0$ on $[r_1, b]$. Now, $\operatorname{sgn}[r(x)\,y''(x)]' = \operatorname{sgn}[r(x)\,y''(x)]'' \neq \operatorname{sgn} y(x)$ on (r_1, b). By (5.10), $y'(b)\,[r(x)\,y''(x)]'_{x=b} > 0$.

Suppose (ii) holds. If $y'(a)\,y''(a) \geqslant 0$, we are done by the above discussion. Assume then that

$$y'(a)\,y''(a) < 0. \tag{5.12}$$

For $\delta > 0$ and sufficiently small, $y(x)\,y'(x) > 0$ on the interval $(a, a+\delta)$. Let $\bar{a} > a$ such that $y(\bar{a}) = 0$ and $y(x) \neq 0$ on $(a, \bar{a})$. As before,

$$\operatorname{sgn} y(x) \neq \operatorname{sgn}[r(x)\,y''(x)]'', \qquad \text{for} \qquad x \in (a, \bar{a}).$$

By (5.12),

$$y(x)\,y''(a) < 0 \qquad \text{for} \qquad x \in (a, \bar{a}). \tag{5.13}$$

Consequently, $y''(x)\,[r(x)\,y''(x)]' \neq 0$ for $x \in (a, \bar{a})$, by (5.13) and (ii). Since $y'(a)\,y'(\bar{a}) < 0$, by (5.12) and (5.13), $y'(\bar{a})\,y''(\bar{a}) > 0$. Applying (i) of Theorem 5.2, the proof of the theorem is complete.

The following theorem describes another type of behavior of oscillatory solutions.

THEOREM 5.3. *Suppose $y(x)$ is a solution of equation* (5.1) *such that for some $x = a \geqslant \epsilon$, $y(a) = 0$. Suppose further that*

$$y'(a)\,y''(a)\,[r(x)\,y''(x)]'_{x=a} \neq 0$$

and

$$\operatorname{sgn} y'(a) = \operatorname{sgn}[r(x)\,y''(x)]'_{x=a} \neq \operatorname{sgn} y''(a).$$

If $b > a$ such that $y(b) = 0$ and if

$$\int_{\epsilon}^{\infty} [r(t)\,y''^2(t) + p(t)\,y^2(t)]\,dt \leqslant -r(a)\,y'(a)\,y''(a), \tag{5.14}$$

then

$$y'(b)\,y''(b)\,[r(x)\,y''(x)]'_{x=b} \neq 0, \tag{5.15}$$

and

$$\operatorname{sgn} y'(b) = \operatorname{sgn}[r(x)\,y''(x)]'_{x=b} \neq \operatorname{sgn} y''(b). \tag{5.16}$$

Proof. By (5.14), $F[y(x)] < 0$ for $x \geqslant a$. Then

$$r(b)\, y'(b)\, y''(b) = F[y(b)] < 0.$$

Hence

$$\operatorname{sgn} y'(b) \neq \operatorname{sgn} y''(b). \tag{5.17}$$

First, suppose $y(x) \neq 0$ on (a, b). Without loss of generality, we may assume that $y'(a) > 0$. Then, $y'(b) < 0$ and, by (5.17), $y''(b) > 0$. Hence, $y'(x)$ and $y''(x)$ vanish an odd number of times on (a, b). Using this fact and Rolle's Theorem, a simple argument shows that $y'(x)$ and $y''(x)$ vanish exactly once on (a, b).

We now show that $[r(x)\, y''(x)]'$ vanishes exactly once on (a, b). Rolle's Theorem guarantees that $[r(x)\, y''(x)]'$ does not vanish more than once on (a, b). Suppose $[r(x)\, y''(x)]' \neq 0$ on (a, b). Since $y'(a) > 0$, we have by hypothesis that $[r(x)\, y''(x)]' > 0$ on (a, b). Now $y(x) > 0$ on (a, b). Let $\bar{b} > b$ such that $y(\bar{b}) = 0$ and $y(x) \neq 0$ on $(b, \bar{b})$. Then $y(x) < 0$ on $(b, \bar{b})$. So $[r(x)\, y''(x)]'' = -p(x)\, y(x) > 0$ on $(b, \bar{b})$, and, consequently, $[r(x)\, y''(x)]' > 0$ on $(b, \bar{b})$. Therefore, $y''(x) > 0$ on $[b, \bar{b}]$. Since $y'(b) < 0$, we have $y'(\bar{b}) > 0$. So $y'(\bar{b})\, y''(\bar{b}) > 0$. But this contradicts (5.17). Hence, $[r(x)\, y''(x)]'$ has exactly one zero on (a, b). Therefore, (5.15) and (5.16) hold for $x = b$, if b is the first zero of $y(x)$ following $x = a$.

Suppose now that $x = b$ is not the next zero of $y(x)$ following $x = a$. Since there is only a finite number of zeros of $y(x)$ on (a, b), an induction argument will show that (5.15) and (5.16) hold at $x = b$.

The following lemma shows that condition (5.14) is also necessary for this second type of oscillatory behavior.

LEMMA 5.1. *Let $y(x)$ be a solution of Eq.* (5.1) *such that for some $a \geqslant \epsilon$, $y(a) = 0$,*

$$y'(a)\, y''(a)\, [r(x)\, y''(x)]'_{x=a} \neq 0,$$

and

$$\operatorname{sgn} y'(a) = \operatorname{sgn}[r(x)\, y''(x)]'_{x=a} \neq \operatorname{sgn} y''(a).$$

If for some point $x = b > a$,

$$\int_a^b [r(t)\, y''^2(t) + p(t)\, y^2(t)]\, dt \geqslant -r(a)\, y'(a)\, y''(a),$$

then for each $c > b$ for which $y(c) = 0$, we have that (5.8) *and* (5.9) *hold.*

Proof. Let $c > b$ such that $y(c) = 0$. Then,

$$r(c)\, y'(c)\, y''(c) = F[y(c)] > F[y(b)] \geqslant 0.$$

Hence, $y'(c)\, y''(c) > 0$, and, by Theorem 5.2, the proof of the lemma is complete.

We summarize the results of Theorems 5.2 and 5.3 and Lemma 5.1 in the following theorem.

THEOREM 5.4. *Let $y(x)$ be a solution of Eq.* (5.1). *There then exists a number $a \geqslant \epsilon$ such that if $b > a$ with $y(b) = 0$, then*

$$y'(b)\, y''(b)\, [r(x)\, y''(x)]'_{x=b} \neq 0$$

and either

(i) $\operatorname{sgn} y'(b) = \operatorname{sgn}[r(x)\, y''(x)]'_{x=b} \neq \operatorname{sgn} y''(b)$,

or

(ii) $\operatorname{sgn} y'(b) = \operatorname{sgn} y''(b) = \operatorname{sgn}[r(x)\, y''(x)]'_{x=b}$.

Furthermore, $y(x)$ has property (i) *for every point $x = b$ for which $y(b) = 0$ if and only if*

$$\int_c^{\infty} [r(t)\, y''^2(t) + p(t)\, y^2(t)]\, dt \leqslant -r(c)\, y'(c)\, y''(c), \tag{5.18}$$

where c is the first zero of $y(x)$ to the right of ϵ.

In order to simplify the statements of the following theorems we make the following definitions. Let $y(x)$ be a solution of Eq. (5.1). Then $y(x)$ is said to be a *Type* I-*solution* provided for each zero of $y(x)$, say $x = b$, we have $y'(b)\, y''(b)\, [r(x)\, y''(x)]'\ x = b \neq 0$ and condition (i) of Theorem 5.4 holds. A nontrivial solution of Eq. (5.1) is called a *Type* II-*solution* provided it is not Type I. Such a solution will eventually satisfy condition (ii) of Theorem 5.4.

Every nontrivial solution $y(x)$ of (5.1) is either of Type I or Type II. In either case, Theorem 5.4 says that between consecutive zeros of $y(x)$, the functions $y'(x)$, $y''(x)$, and $[r(x)\, y''(x)]'$ have the same number of zeros. (We are assuming, of course, that if $y(x)$ is of Type II, the zeros of $y(x)$ are large enough.) Since $y'(x)$ vanishes at least once and $[r(x)\, y''(x)]'$ cannot vanish more than once, each function $y'(x)$, $y''(x)$, $[r(x)\, y''(x)]'$ has exactly one zero between the consecutive zeros of $y(x)$. We have then the following corollary.

COROLLARY 5.1. *Let $y(x)$ be a solution of* (5.1). *Let $a < b$ be consecutive zeros of $y(x)$, and let r_1, r_2, r_3 denote the zeros of $y'(x)$, $y''(x)$, $[r(x)\, y''(x)]'$, respectively, in the interval (a, b). Then*

(i) *if* $y(x)$ *is* Type I, $a < r_1 < r_2 < r_3 < b$,

and

(ii) *if* $y(x)$ *is* Type II *and* $x = a$ *is large enough, then* $a < r_3 < r_2 < r_1 < b$.

Proof. By Theorem 5.4 and the fact that the functions $y'(x)$, $y''(x)$, $[r(x)\,y''(x)]'$ vanish exactly once on (a, b), we have (ii) holding. For the same reasons, if $y(x)$ is of Type I, we have

$$a < r_1 \leqslant r_2 \leqslant r_3 < b. \tag{5.19}$$

However, it is obvious that if either equality in (5.19) holds, then $y'(b)\,y''(b) > 0$ or $y''(b)\,[r(x)\,y''(x)]'_{x=b} > 0$. This contradicts the fact that $y(x)$ is a Type I-solution. Hence (i) holds if $y(x)$ is of Type I.

It is an immediate consequence of Lemmas 9.2 and 9.3 in Leighton and Nehari's paper [7] that the zeros of the principal solution at $x = a$ and an essentially different solution $y(x)$ with $y(a) = 0$ will separate each other on (a, ∞), if $y(x)$ is of Type II, and on $[\epsilon, a)$, if $y(x)$ is of Type I.

We next turn our attention to the question of the existence of solutions of Type I and Type II. Certainly there exists solutions of Type II. For example, the principal solution at $x = a$ is a Type II-solution. In order to show Type I-solutions exist, we shall use the following lemma.

LEMMA 5.2. *Suppose* $y(x)$ *is a solution of* (5.1). *If there exists a number* $a \in [\epsilon, \infty)$ *for which* $y(a) = 0$ *and conditions* (5.15) *and* (5.16) *hold, then for any* $b \in [\epsilon, a]$ *for which* $y'(b)\,y''(b)\,[r(x)\,y''(x)]'_{x=b} \neq 0$, *the numbers* $y'(b)$, $y''(b)$, *and* $[r(x)\,y''(x)]'_{x=b}$ *cannot all have the same sign.*

Proof. Suppose there exists a number $b \in [\epsilon, a)$ for which $y'(b) > 0$, $y''(b) > 0$, $[r(x)\,y''(x)]'_{x=b} > 0$. Then $b < a$ implies $F[y(b)] < F[y(a)] < 0$. So $0 > F[y(b)] = r(b)\,y'(b)\,y''(b) - y(b)\,[r(x)\,y''(x)]'_{x=b}$ which implies $y(b) > 0$. Let $c \in (b, a]$ such that $y(c) = 0$ and $y(x) \neq 0$ on $[b, c)$. Now as in the discussion preceding Corollary 5.1, we have $y'(x)$, $y''(x)$, $[r(x)\,y''(x)]'$ vanishing exactly once on (b, c). Hence $y'(c)\,y''(c) > 0$, and by Theorem 5.2, $y'(a)\,y''(a) > 0$, a contradiction. Therefore no such point $x = b$ exists.

THEOREM 5.5. *There exists a Type* I*-solution of Eq.* (5.1).

Proof. For each natural number $n \geqslant 1 + \epsilon$ let $u_n(x)$ denote a nontrivial solution of (5.1) such that

$$u_n(n) = u_n'(n) = u_n(n-1) = 0. \tag{5.20}$$

Then $u_n(x)$ is a Type II-solution. Since $u_n(x)$ has a double zero at $x = n$, Theorem 5.2 implies $u_n'(n-1)\,u_n''(n-1)\,[r(x)\,u_n''(x)]'_{x=n-1} \neq 0$ and

$$\operatorname{sgn} u_n'(n-1) = \operatorname{sgn}[r(x)\,u_n''(x)]'_{x=n-1} \neq \operatorname{sgn} u_n''(n-1). \tag{5.21}$$

By Lemma 5.2, for $b \in [\epsilon, n-1]$ with

$$u_n'(b)\, u_n''(b)\, [r(x)\, u_n''(x)]'_{x=b} \neq 0, \tag{5.22}$$

the values $u_n'(b)$, $u_n''(b)$, and $[r(x)\, u_n''(x)]'_{x=b}$ cannot have the same sign.

Now let $z_1(x)$, $z_2(x)$, $z_3(x)$, and $z_4(x)$ be four linearly independent solutions of (5.1). Then for each n, there exist constants c_{n1}, c_{n2}, c_{n3}, c_{n4} such that

$$u_n(x) = c_{n1}z_1(x) + c_{n2}z_2(x) + c_{n3}z_3(x) + c_{n4}z_4(x) \tag{5.23}$$

and

$$c_{n1}^2 + c_{n2}^2 + c_{n3}^2 + c_{n4}^2 = 1. \tag{5.24}$$

Then each of the sequences

$$\{c_{n1}\}_{n=1}^{\infty}, \qquad \{c_{n2}\}_{n=1}^{\infty}, \qquad \{c_{n3}\}_{n=1}^{\infty}, \qquad \text{and} \qquad \{c_{n4}\}_{n=1}^{\infty}$$

is bounded. So for each $k = 1, 2, 3, 4$ there exists a convergent subsequence

$$\{c_{n_j k}\}_{j=1}^{\infty}.$$

Define $c_k = \lim_{j\to\infty} c_{n_j k}$ for $k = 1, 2, 3, 4$. Let

$$w(x) = c_1z_1(x) + c_2z_2(x) + c_3z_3(x) + c_4z_4(x).$$

By (5.24), $c_1^2 + c_2^2 + c_3^2 + c_4^2 = 1$. Hence, $w(x) \not\equiv 0$. Then, by (5.23), the sequences

$$\{u_{n_j}(x)\}_{j=1}^{\infty}, \qquad \{u'_{n_j}(x)\}_{j=1}^{\infty}, \qquad \{u''_{n_j}(x)\}_{j=1}^{\infty}, \qquad \text{and} \qquad \{[r(x)\, u''_{n_j}(x)]'\}_{j=1}^{\infty}$$

converge uniformly on any finite subinterval of $[\epsilon, \infty)$ to the functions $w(x)$, $w'(x)$, $w''(x)$, and $[r(x)\, w''(x)]'$, respectively. Since all solutions of (5.1) are oscillatory, $w(x)$ is oscillatory, and by Theorem 5.4, $w(x)$ is either of Type I or of Type II.

We shall show $w(x)$ is not of Type II. Suppose the contrary. Then there exists a number $b \in [\epsilon, \infty)$ such that $w(b) = 0$, $w'(b)\, w''(b)\, [r(x)\, w''(x)]'_{x=b} \neq 0$, and $\operatorname{sgn} w'(b) = \operatorname{sgn} w''(b) = \operatorname{sgn}[r(x)\, w''(x)]'_{x=b}$. But this would mean that for j large enough, $\operatorname{sgn} u'_{n_j}(b) = \operatorname{sgn} u''_{n_j}(b) = \operatorname{sgn}[r(x)\, u''_{n_j}(x)]'_{x=b}$, which contradicts previous observations. Hence $w(x)$ is a Type I-solution.

The following two theorems are analogus to the Bôcher-Osgood theorem [6] for second order equations.

THEOREM 5.6. *Suppose $y(x)$ is a Type* I*-solution of Eq.* (5.1). *If $p \in C'$ and $p'(x) \geqslant 0$, then $|y(x)|$ is decreasing at its successive maxima and minima.*

Proof. Since $y(x)$ is a Type I-solution, for each $a \geqslant \epsilon$ for which $y(a) = 0$, we have

$$\operatorname{sgn} y'(a) = \operatorname{sgn}[r(x)\, y''(x)]'_{x=a} \neq \operatorname{sgn} y''(a).$$

Let $\epsilon \leqslant t_0 < t_1 < t_2$ be three consecutive zeros of $y(x)$. By Corollary 5.1, there exist numbers $r_1, r_2, r_3, s_1, s_2, s_3$ such that

$$t_0 < r_1 < r_2 < r_3 < t_1 < s_1 < s_2 < s_3 < t_2$$

and

$$y'(r_1) = y'(s_1) = y''(r_2) = y''(s_2) = 0$$

and

$$[r(x)\, y''(x)]'_{x=r_3} = [r(x)\, y''(x)]'_{x=s_3} = 0$$

and

$$y'(x)\, y''(x)\, [r(x)\, y''(x)]' \neq 0$$

at any other point of $[t_0, t_2]$. Assume without loss of generality that $y'(t_0) > 0$. We wish to show $| y(s_1)| < | y(r_1)|$.

Consider the following identity, where $a \geqslant \epsilon$.

$$\begin{aligned} \bar{G}[y(x)] &\equiv \frac{[r(x)\, y''(x)]'\, y'(x)}{p(x)} + \frac{1}{2} y^2(x) \\ &= \bar{G}[y(a)] + \int_a^x \frac{[r(t)\, y''(t)]'}{p(t)} \left[y''(t) - \frac{y'(t)\, p'(t)}{p^2(t)} \right] dt. \end{aligned} \tag{5.25}$$

Let $x = s_1$ and $a = r_3$. On the interval (r_3, s_1), $[r(t)\, y''(t)]' < 0$, $p(t) > 0$, $y''(t) > 0$, $p'(t) \geqslant 0$, and $y'(t) < 0$. Consequently, $\bar{G}[y(x)]$ is strictly decreasing on (r_1, s_3). So

$$\tfrac{1}{2} y^2(r_3) = \bar{G}[y(r_3)] > \bar{G}[y(s_1)] = \tfrac{1}{2} y^2(s_1),$$

and

$$| y(r_3)| > | y(s_1)|. \tag{5.26}$$

On the interval (r_1, r_3), $y(x)\, y'(x) < 0$. Thus $y^2(r_3) < y^2(r_1)$, and $| y(r_3)| < | y(r_1)|$. Hence, by (5.26), $| y(r_1)| > | y(s_1)|$, and the proof is complete.

The above theorem is independent of additional conditions on $r(x)$. The following theorem is the companion to Theorem 5.6 for Type II-solutions. The proof is analogous to that of Theorem 5.6.

THEOREM 5.7. *Let $y(x)$ be a Type* II-*solution of Eq.* (5.1). *If $p'(x) \leqslant 0$, then $| y(x)|$ is increasing at consecutive maxima and minima.*

6. *Properties of Nonoscillatory Solutions of* (1.4)

In this section we shall assume Eq. (5.1) has a nonoscillatory solution. Under this assumption it has been shown [7] that all the solutions of (5.1) are nonoscillatory. We begin with the following theorem.

THEOREM 6.1. *Let $y(x)$ be a nonoscillatory solution of equation* (5.1). *Then for large x*

$$y(x)\, y'(x)\, y''(x)\, [r(x)\, y''(x)]' \neq 0$$

and $y(x)$ satisfies one of the following four conditions:

(i) $\operatorname{sgn} y(x) = \operatorname{sgn} y'(x) = \operatorname{sgn} y''(x) = \operatorname{sgn}[r(x)\, y''(x)]'$

(ii) $\operatorname{sgn} y(x) = \operatorname{sgn} y'(x) = \operatorname{sgn}[r(x)\, y''(x)]' \neq \operatorname{sgn} y''(x)$

(iii) $\operatorname{sgn} y(x) = \operatorname{sgn} y''(x) = \operatorname{sgn}[r(x)\, y''(x)]' \neq \operatorname{sgn} y'(x)$

(iv) $\operatorname{sgn} y(x) = \operatorname{sgn} y'(x) \neq \operatorname{sgn} y''(x) = \operatorname{sgn}[r(x)\, y''(x)]'$.

Proof. If $y(x)\, [r(x)\, y''(x)]' > 0$ for x sufficiently large, then it is obvious that one of the conditions (i)-(iii) holds. So assume that $y(x)\, [r(x)\, y''(x)]' < 0$ for x sufficiently large. Then eventually

$$[r(x)\, y''(x)]'\, [r(x)\, y''(x)]'' > 0. \tag{6.1}$$

Consequently, for large x

$$y''(x)\, [r(x)\, y''(x)]' > 0 \tag{6.2}$$

and so

$$y''(x)\, y(x) < 0. \tag{6.3}$$

If $y'(x)\, y''(x) > 0$, then for x large enough

$$y(x)\, y''(x) > 0. \tag{6.4}$$

But (6.4) contradicts (6.3). Therefore

$$y'(x)\, y''(x) < 0, \tag{6.5}$$

and (iv) holds. The proof is then complete.

In the next two theorems we give some necessary conditions under which solutions satisfying (i)-(iv) of Theorem 6.1 may exist.

THEOREM 6.2. *Suppose $p'(x)$ exists and does not change sign for large x. If $y(x)$ is a solution of* (5.1) *which satisfies either* (i) *or* (ii) *of Theorem 6.1, then $p'(x)$ is negative for large x.*

Proof. Assume $p'(x) \geqslant 0$ eventually, and suppose $y(x) > 0$ for x sufficiently large. If $y(x)$ satisfies either (i) or (ii), then

$$y'(x) > 0, \qquad [r(x)\, y''(x)]' > 0. \tag{6.6}$$

But for x large enough,

$$[r(x)\, y''(x)]'' = -\, p(x)\, y(x) < 0 \tag{6.7}$$

and

$$[r(x)\, y''(x)]''' = -\, [p'(x)\, y(x) + p(x)\, y'(x)] < 0. \tag{6.8}$$

Now (6.7) and (6.8) imply $[r(x)\, y''(x)]' < 0$ eventually. This last inequality contradicts (6.6), and the proof is complete.

THEOREM 6.3. *Suppose $r'(x)$ exists and does not change sign for large x. If $y(x)$ is a solution of* (5.1) *which satisfies either* (iii) *or* (iv) *of Theorem 6.1, then $r'(x)$ is eventually positive.*

Proof. Suppose $r'(x) \leqslant 0$ for large x. If $y(x)$ satisfies (iii) or (iv), then for x sufficiently large

$$\operatorname{sgn} y'(x) \neq \operatorname{sgn} y''(x) = \operatorname{sgn}[r(x)\, y''(x)]'. \tag{6.9}$$

Now $[r(x)\, y''(x)]' = r'(x)\, y''(x) + r(x)\, y'''(x)$. Consequently, by the assumption on $r'(x)$,

$$\operatorname{sgn} y'''(x) = \operatorname{sgn} y''(x). \tag{6.10}$$

But (6.10) implies $\operatorname{sgn} y''(x) = \operatorname{sgn} y'(x)$. This fact contradicts (6.9), and the proof is complete.

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